

# Stock Price Processes with Infinite Source Poisson Agents

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## Abstract

We construct a general stochastic process and prove weak convergence results. It is scaled in space and through the parameters of its distribution. We show that our simplified scaling is equivalent to time scaling used frequently. The process is constructed as an integral with respect to a Poisson random measure which governs several parameters of trading agents in the context of stock prices. When the trading occurs more frequently and in smaller quantities, the limit is a fractional Brownian motion. In contrast, a stable Lévy motion is obtained if the rate of trading decreases while its effect rate increases.

*Key words:* fractional Brownian motion, arbitrage, stock price model, stable Lévy motion, long-range dependence, self-similarity

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## 1. Introduction

Weak convergence of scaled input processes has been studied extensively over the last decade [3, 10, 15, 18, 22, 25, 27, 28, 29, 31, 32, 35, 36]. The limit is a fractional Brownian motion (fBm) or a Lévy process depending on the particular scaling. While the motivation of such analysis originates from data traffic in telecommunications, both fBm and Lévy processes have recently become prevalent in finance. Thereby, we construct a general stochastic process based on a Poisson random measure  $N$ , interpret it as stock price process and prove weak convergence results.

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We consider a real valued process of the form

$$Z_n(t) = \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} a_n r u \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] N_n(ds, du, dr) \quad (1)$$

where  $f$  is a deterministic function satisfying Lipschitz condition,  $(a_n)$  is a scaling sequence, and  $r$  and  $u$  are marks of the resulting Poisson point process with  $s$  denoting the time. The mean measure on  $U$  is either a probability measure or an infinite measure on  $\mathbb{R}_+$ . The process  $Z_n$  depends on the scaling parameter  $n \in \mathbb{R}_+$  through not only  $a_n$ , but also through the mean measure  $\mu_n$  of  $N_n$  which is taken to have a regularly varying form in  $u$  to comply with the long-range dependence property of teletraffic or financial data. After centering the process  $Z_n$ , we can obtain either an fBm or a stable Lévy motion depending on the particular scaling of the mean measure of  $N_n$  and the factor  $a_n$  as  $n \rightarrow \infty$ . While fBm is a self-similar and long-range dependent model, a Lévy process has independent increments and self-similarity exists without long-range dependence.

Our main contribution is the generalization of the previous results [25, 31] with a specific linear form for  $f$  and [29] with an increasing  $f$  satisfying some other technical conditions, to those with Lipschitz functions. In this case, not only the proofs require more work, but we need Lipschitz assumptions on the derivative of  $f$  as well. We also show that the time scaling used in previous work can be replaced by parameter scaling of the distributions of the relevant random variables. The time scaling has been interpreted as ‘birds-eye’ description of a process, which is not necessary when the scaling is interpreted in terms of its parameters. Inspired by [9], we unify the results for general forms of  $f$  with less stringent conditions in some cases. In [9], it is noted that a fractional Brownian motion with  $H > 1/2$  can be approximated if the pulse  $f$  is continuous on  $\mathbb{R}$  and has a compact support. As an alternative extension, we consider continuous  $f$  with no compact support while constructing (1) with  $\tilde{N}_n = N_n - \mu_n$  as a centered process. A secondary generalization of previous work is the consideration of  $Z$  as a more general process than workload which would be positive by definition. We allow a signed process through the choice of real valued rate  $r$ .

In related work [15, 32], the Poisson random measure is replaced by a general

arrival process and a cluster Poisson process, respectively. Ergodicity is required for the limit theorems in the general arrival case as well. On the other hand, most of the previous studies on scaled input processes are named infinite source Poisson models due to the assumption of Poisson arrivals.

As for application in finance, the process  $Z_n$  can be interpreted as the price of a stock. The interpretation of (1) as a stock price process has been first presented in [1, 11]. Our aim is to construct a model involving the behavior of agents that can be parameterized and estimated from data, yet having well-known stochastic processes as its limits. While the limiting models fit well to financial data, they do not involve the physical parameters of the trading agents. Agent based modeling is widely used to find a model that best fits stock price processes. In some studies, agents are divided into two groups, mostly named as chartists and fundamentalists. In these studies, the two agent groups have different demand functions for the stock [14, 21, 23] and the price is generally determined via the total excess demand [7, 8, 13, 20, 34]. In (1), the arrival time  $s$ , the rate  $r$  and the duration  $u$  of the effect of an order are all governed by the Poisson random measure. Under the assumption of positive correlation between the total net demand and the price change, we expect that a buy order of an agent increases the price whereas a sell order decreases it. Each order has an effect proportional to its volume and duration. The duration of the effect is assumed to follow a heavy tailed distribution. This effect starts when the order is given, increases to a maximum which is proportional to the total order amount, and then starts decreasing until it vanishes after a finite time. Alternatively, its effect may last for some time and leave the price at a changed level on and after the time of transaction. The logarithm of the stock price is found by aggregating the incremental effects of orders placed by all active agents in  $[0, t]$ . As a semi-martingale, our process does not allow for arbitrage. It is a novel model which is alternative to existing agent based constructions that are semi-Markov processes [3].

We prove weak convergence results for the two different measures used for the duration, separately. As for their connection, the infinite measure being the limit of the probability measure is proved in Theorem 1. Although a probability measure is appropriate for the duration, its limiting form is able to capture the essential

tail behavior which represents long-range dependence and self-similarity properties. Moreover, the scalings with the limiting measure is simpler to interpret as follows. An fBm limit is found as the frequency of trading increases and the price effect of the orders decreases as shown in Theorem 3. In contrast, a stable process which is a scaled version of a stable Lévy motion is obtained in Theorem 5 when the rate of trading decreases and the effect of the orders increases. Theorems 2 and 4 are concerned with fBm and stable limits, respectively, for the case of probability measure for the duration. The hypotheses in these theorems involve extra scaling needed for obtaining the limiting measure in addition to the scaling of the rates and effects. The stable process obtained in the limit has a skewness parameter that depends on the distribution of the rate  $r$  as it can take positive or negative values.

The paper is organized as follows. In Section 2, we define the ingredients of the workload model, namely the random variables associated with the process to be constructed. The price process is defined in Section 3 with various assumptions on the parameters of the Poisson random measure. Section 4 includes limit theorems for fractional Brownian motion. Finally, limit theorems for stable Lévy motion are proved in Section 5.

## 2. Infinite Source Poisson Agents

In this section, we state our main assumptions and notation for constructing a stock price process. We assume potentially an infinite pool of agents and gather the agents' trading processes under a Poisson random measure. Each agent's trading starts according to the underlying Poisson process and it ends after a random amount of time. Although the same agents may be returning for further transaction, new arrivals are assumed to be independent and identically distributed.

The agents are called *infinite source Poisson* as a term borrowed from traffic modeling in telecommunications based on Poisson random measures (e.g. [10, 31, 32]). Since self-similarity and long-range dependence are common statistical properties observed in both Internet traffic and financial data, the stochastic models also have similar features.

In [3], a finite number of identically distributed semi-Markov processes representing the trading states of the agents over time are aggregated to form the price process. The number of semi-Markov processes, equivalently, the number of agents has been taken to infinity only in the limit. On the other hand, our price model is a stationary process with an infinite source of agents that arrive according to a Poisson process. We concentrate on scaling the other parameters that have physical interpretations for obtaining the limiting stochastic processes.

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. Let  $\mathcal{B}_{\mathbb{R}}$  denote the Borel  $\sigma$ -algebra on  $\mathbb{R}$ . Let  $N$  be a Poisson random measure on  $(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}, \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{B}_{\mathbb{R}})$  with mean measure

$$\mu(ds, du, dr) = \lambda ds \nu(du) \gamma(dr) \quad (2)$$

where  $\lambda > 0$  is the arrival rate of the underlying Poisson process,  $1 < \delta < 2$ ,  $\gamma$  is the distribution of a random variable  $R$  and  $\nu$  is either a probability measure that satisfies

$$\int_u^\infty \nu(dy) \sim h(u) \frac{u^{-\delta}}{\delta} \quad \text{as } u \rightarrow \infty \quad (3)$$

where  $h$  is a slowly varying function at infinity, that is,  $h$  is such that

$$\lim_{n \rightarrow \infty} h(un)/h(n) = 1 \quad (4)$$

or a measure given by

$$\nu(du) = u^{-\delta-1} du. \quad (5)$$

Each atom  $(S_j, U_j, R_j)$  of  $N$  can be interpreted as an order from an agent, where  $S_j$  is the arrival time of the order,  $U_j$  is the duration of its effect on the price, and  $R_j$  denotes its rate which also plays the role of conversion to monetary units. The sign of  $R_j$  could be positive or negative depending on the order being a buy or sell order, respectively. Under assumption (5) for the measure  $\nu$ , the duration  $U$  is obtained from a diffuse measure on  $(0, \infty)$  with no probability distribution. This case is studied for suppressing the less significant details in the proofs of the convergence theorems. In case of (3),  $U$  follows a heavy tailed distribution with finite mean but infinite variance, and the convergence proofs involve the function  $h$ . Although the latter case is physically more meaningful, the scalings are more involved as well in the limit theorems for the self-similar processes fBm and stable Lévy motion.

Let  $K : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$  denote an effect function such that  $K(x, \cdot, \cdot) \equiv 0$  if  $x < 0$ . The effect of an order starting at  $s$  and ending at  $s + u$  depends on the rate  $r$  of the effect and equals  $K(t - s, u, r)$  at time  $t$  provided that  $s \leq t$ . The function  $K$  can also be interpreted as the local dynamics of a transaction as a result of a buy or sell order. The rate  $r$  will be connected to the quantity of the order which will be elaborated further in the sequel. We specify a general effect function  $f : \mathbb{R} \rightarrow \mathbb{R}$  that determines  $K$  by

$$K(t - s, u, r) = ru f\left(\frac{t - s}{u}\right) \quad (6)$$

for  $t \geq s$ , and  $f(x) = 0$  for  $x < 0$ . We consider  $f$  to be the deterministic shape function, or pulse, for the effect which is then shifted to the starting position  $s$ , scaled and amplified for the duration  $u$ , and adjusted once more with the rate/conversion factor  $r$ , all randomized by  $N$ .

The price process will be constructed as a sum of randomized pulses. Characterization of  $f$  that will yield an fBm or a Lévy motion for the price process is of interest, provided that the parameters in (2) are appropriately scaled. In the workload processes studied in [25, 29, 31, 32],  $f$  has the following form

$$f(x) = \begin{cases} x \wedge 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (7)$$

which represents an increasing input, or effect, with unit rate on  $[0, 1]$  and remains constant thereafter. In [9], the pulse

$$f(x) = \begin{cases} 1/2 - |x - 1/2| & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

is considered for the aim of approximating an fBm. It has a compact support representing a limited effect that vanishes after the duration of the pulse. These special pulses, in other words effect functions, are sketched in Fig.1. General Lipschitz functions are also considered in [9].

The effect function is left unspecified in [22, 27, 28] but with conditions on the tail properties of its distribution for large times. The duration is not parameterized in contrast to the present work. In [28], the effect may last indefinitely although it

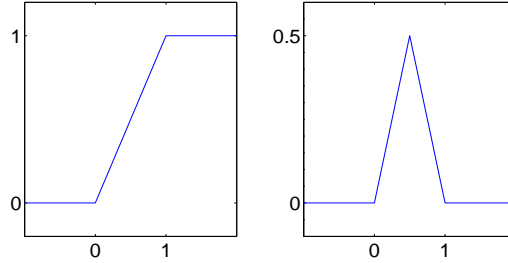


Figure 1: Sample pulses of different kind

decreases in time with a regularly varying tail. In [22], each effect is assumed to converge for large times to a finite random variable which has a distribution with a regularly varying tail. The stable limits are outlined in [27] with several interesting special cases. In [29], the effect function is deterministic which is randomized through a random variable for duration as in our case, but with an additional assumption that the effect function itself also has a regularly varying tail. On the other hand, random effect functions have been also considered also in [30] where central limit theorems are proved under general conditions. As a special case, the effect function could be a compound Poisson process as in [10, 25]. This could be used to model the buy or sell transactions in smaller quantities for a given order in the present work. A semimartingale is assumed for the rate of the effect in [3]. In applications,  $K$  can be estimated to match the local dynamics of the price change or the workload.

### 3. Price Process

Let the price process be given by  $Y = \{Y(t) : Y(t) = \exp Z(t), t \in \mathbb{R}_+\}$  where  $Z$  is the log-price process to be constructed in this section. We aim to introduce a stochastic process which is sufficiently general to approximate an fBm or a Lévy motion, and has an adequate number of physical parameters that can be estimated from data. The effect function and the Poisson random measure described in Section 2 will be the main ingredients.

**Remark 1.** Previous models that involve heterogeneous agents usually classify them

into two separate groups as chartists and fundamentalists according to their trading behavior [3]. This can clearly be generalized to several types of agents. The total effect  $Z^i$  from agents of type  $i$ ,  $i = 1, \dots, I$ , can be further aggregated to form  $Z$  as

$$Z(t) = \sum_{i=1}^n Z^i(t) \quad t \geq 0.$$

Then, the price  $Y$  at time  $t$  is given by  $Y(t) = e^{Z(t)}$  as before.

We form the log-price process  $Z$  by aggregating the randomized effects. More precisely, the difference in the effect amplitudes at times 0 and  $t$  are integrated with respect to the Poisson random measure  $N$  to yield

$$Z(t) = \int_{-\infty}^t \int_0^\infty \int_{-\infty}^\infty (K(t-s, u, r) - K(-s, u, r)) N(ds, du, dr) \quad t \geq 0.$$

Since the underlying Poisson process has been going on long before time 0,  $Z$  has stationary increments and  $Z(0) = 0$  by construction. We think  $Z(t) - Z(0)$  as the sum of all effects due to all active agents between times 0 and  $t$ . We assume that  $K$  has the form (6) as before, and write

$$Z(t) = \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty ru \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] N(ds, du, dr). \quad (9)$$

The propositions below give the sufficient conditions for  $Z$  to be well-defined for a finite measure and a specific  $\sigma$ -finite measure  $\nu$  in Equation (2), respectively. Let  $\phi$  denote the characteristic function of  $Z$ , that is,  $\phi(\xi) = \mathbb{E}e^{i\xi Z}$  for  $\xi \in \mathbb{R}$ .

**Proposition 1.** *Suppose that  $\nu$  is a probability measure satisfying  $\int uv(du) < \infty$ ,  $\mathbb{E}|R| < \infty$  and  $f$  is a Lipschitz continuous function on  $\mathbb{R}$  with  $f(x) = 0$  for all  $x \leq 0$  and  $f(x) = f(1)$  for all  $x \geq 1$ . Then,  $Z(t)$ ,  $t \geq 0$ , is a finite random variable a.s. with characteristic function*

$$\phi(\xi) = \exp \int_{\mathbb{R}} \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left\{ \exp \left[ i\xi ru \left( f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right) \right] - 1 \right\} \lambda ds \nu(du) \gamma(dr)$$

**Proof:** The integral  $\int h d\mu$  of a deterministic function  $h$  with respect to a Poisson random measure defines a finite random variable if  $\int |h \wedge 1| d\mu < \infty$  where  $\mu$  is the mean measure. This is clearly satisfied if  $\int |h| d\mu < \infty$  which also implies that the random variable has a finite expectation [26]. Therefore, it is sufficient to show that the expression

$$I := \int_{-\infty}^\infty \int_0^\infty u \left| f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right| \nu(du) ds$$

is finite for  $Z$  to be well defined as  $\mathbb{E}|R| < \infty$ . Note that

$$I = \int_{-\infty}^0 \int_0^\infty u \left| f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right| \nu(du) ds + \int_0^t \int_0^\infty u \left| f\left(\frac{t-s}{u}\right) \right| \nu(du) ds$$

Considering the two different regions  $\{u : 0 < -s/u < 1 < (t-s)/u\}$  and  $\{u : 0 < -s/u < (t-s)/u < 1\}$  for the first integral, and the regions  $\{u : (t-s)/u > 1\}$  and  $\{u : (t-s)/u < 1\}$  for the second integral in  $I$ , we get

$$\begin{aligned} I &= \int_{-\infty}^0 \left[ \int_{-s}^{t-s} u \left| f(1) - f\left(\frac{-s}{u}\right) \right| \nu(du) + \int_{t-s}^\infty u \left| f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right| \nu(du) \right] ds \\ &\quad + \int_0^t \left[ \int_0^{t-s} u |f(1)| \nu(du) ds + \int_{t-s}^\infty u \left| f\left(\frac{t-s}{u}\right) \right| \nu(du) \right] ds \end{aligned}$$

Due to the Lipschitz hypothesis on  $f$  and since  $f(0) = 0$ , we have

$$\begin{aligned} I &\leq \int_{-\infty}^0 \left[ \int_{-s}^{t-s} u M \left(1 + \frac{s}{u}\right) \nu(du) + \int_{t-s}^\infty u M \frac{t}{u} \nu(du) \right] ds \\ &\quad + \int_0^t \left[ \int_0^{t-s} u M \nu(du) + \int_{t-s}^\infty u M \frac{t-s}{u} \nu(du) \right] ds \end{aligned} \quad (10)$$

where  $M > 0$ . We apply integration by parts for the inner integrals above. For  $-\infty < s < 0$ , integration by parts yields

$$\int_{-s}^{t-s} (s+u) \nu(du) = tF(t-s) - \int_{-s}^{t-s} F(u) du$$

and we have  $t \int_{t-s}^\infty \nu(du) = t\bar{F}(t-s)$  where  $F$  denotes the cumulative distribution function (cdf) of  $U$  and  $\bar{F} = 1 - F$ . For  $0 < s < t$ , we get

$$\int_0^{t-s} u \nu(du) = (t-s)F(t-s) - \int_0^{t-s} F(u) du$$

by integration by parts and we have  $(t-s) \int_{t-s}^\infty \nu(du) = (t-s)\bar{F}(t-s)$ . Putting all expressions together and using  $\bar{F} = 1 - F$ , we simplify (10) as

$$I \leq M \int_{-\infty}^0 \int_{-s}^{t-s} \bar{F}(u) du ds + M \int_0^t \int_0^{t-s} \bar{F}(u) du ds$$

Changing the order of integration, we get

$$I \leq M \int_0^\infty \int_{-u}^{t-u} \bar{F}(u) ds du = Mt\mathbb{E}U$$

which is finite by hypothesis as  $\mathbb{E}U = \int u\nu(du)$ . The characteristic function of  $Z$  can be found immediately from formulae for integrals with respect to a Poisson random measure [26].  $\square$

The function  $f$  characterized in Proposition 1 represents the local dynamics due to the effect of an individual buy or sell order. The change in price, which can be non-monotonic, occurs over a finite time  $U$  and remains at the same level thereafter. The specific shape of  $f$  is left unspecified, and so is its sign. In general, we expect a buy order to increase the price and a sell order to decrease it. Therefore, if  $f$  is chosen to be an increasing function, then we could have  $R > 0$  for a buy order and  $R < 0$  for a sell order. However, a general form is assumed to leave room for modeling purposes in view of real data and to provide mathematical generality. The special case (7) used in [25, 32, 31] is a linearly increasing pulse as stated in the following corollary.

**Corollary 1.** *Suppose that  $\nu$  is a probability measure satisfying  $\int u\nu(du) < \infty$ ,  $\mathbb{E}|R| < \infty$  and  $f(x) = x \wedge 1$ ,  $x \geq 0$ . Then,  $Z(t)$ ,  $t \geq 0$ , is finite a.s. and its characteristic function is given by*

$$\exp \int_{\mathbb{R}} \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left[ e^{i\xi r(u \wedge (t-s)^+ - u \wedge (-s)^+)} - 1 \right] \lambda ds \nu(du) \gamma(dr)$$

for  $\xi \in \mathbb{R}$ .

**Remark 2.** The log-price process  $Z$  which is a semimartingale in general, becomes a martingale if its mean is zero. This would be satisfied if  $\mathbb{E}R = 0$ , which corresponds to symmetric effects from buy and sell orders, for example.

The following proposition is based on the results of [9]. The support of  $f$  is chosen as  $[0,1]$  for simplicity without loss of generalization.

**Proposition 2.** *Suppose that  $\nu(du) = u^{-\delta-1} du$ ,  $\mathbb{E}R^2 < \infty$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz continuous with compact support  $[0,1]$ . Then,  $Z$  is a well-defined random variable with characteristic function*

$$\exp \int_{\mathbb{R}} \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left\{ e^{i\xi ru[f(\frac{t-s}{u}) - f(\frac{-s}{u})]} - 1 - i\xi ru[f(\frac{t-s}{u}) - f(\frac{-s}{u})] \right\} u^{-\delta-1} \lambda ds du \gamma(dr)$$

**Proof:** The Lipschitz assumption on  $f$  and that it has compact support  $[0,1]$  imply  $f(0) = f(1) = 0$  and

$$\int_{\mathbb{R}} \int_{\mathbb{R}_+} \left[ f(\frac{t-s}{u}) - f(\frac{-s}{u}) \right]^2 u^{1-\delta} du ds < \infty \quad (11)$$

for each  $t > 0$ , and  $1 < \delta < 2$  in particular, by [9, Prop.3.1]. We sketch the usual proof for defining  $Z$  as an almost sure limit of zero mean random variables, as in [9]. Let  $A_k = (2^{-k}, 2^{-k+1}]$ ,  $k = 1, 2, \dots$ ,  $A_0 = (1, \infty)$  be a partition of  $\mathbb{R}_+$ . Clearly,

$$\int_{-\infty}^{\infty} \int_{A_k} \int_{-\infty}^{\infty} r u \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] u^{-\delta-1} \lambda ds du \gamma(dr) = 0,$$

for  $k = 1, 2, \dots$ , due to the form of the effect function and boundedness of  $A_k$ , and also for  $k = 0$  in view of (11). Hence, the random variables

$$\int_{-\infty}^{\infty} \int_{A_k} \int_{-\infty}^{\infty} r u \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] N(ds, du, dr) \quad k = 0, 1, \dots, \quad (12)$$

are well-defined, independent and have zero expectations. One can show that the sum of their variances given by

$$\int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} r^2 u^2 \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right]^2 u^{-\delta-1} \lambda ds du \gamma(dr) \quad (13)$$

is finite using (11) and the assumption that  $\mathbb{E}R^2 < \infty$ . Therefore, the series

$$\sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \int_{A_k} \int_{-\infty}^{\infty} r u \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] N(ds, du, dr)$$

is a.s. convergent by [24, Lemma 3.16]. The limit is denoted by (9) which is a stochastic integral in general since it may be defined by an almost sure limit as above. Its characteristic function at  $\xi \in \mathbb{R}$  is the limit of the characteristic functions of the partial sums of the random variables (12), since almost sure convergence implies convergence in distribution. Now, the characteristic function of (12) can be written as

$$\exp \int_{\mathbb{R}} \int_{A_k} \int_{\mathbb{R}} \left\{ e^{i\xi r u \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right]} - 1 - i\xi r u \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] \right\} u^{-\delta-1} \lambda ds du \gamma(dr)$$

since (12) has zero mean. Using the independence of (12) by disjointness of  $A_k$ ,  $k = 0, 1, 2, \dots$ , we can write the characteristic function of their partial sums up to say  $m \in \mathbb{Z}_+$  as

$$\exp \int_{\mathbb{R}} \int_{2^{-m}}^{\infty} \int_{\mathbb{R}} \left\{ e^{i\xi r u \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right]} - 1 - i\xi r u \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] \right\} u^{-\delta-1} \lambda ds du \gamma(dr)$$

Due to the inequality  $|e^{ix} - 1 - ix| < \frac{1}{2}x^2$  for  $x \in \mathbb{R}$  and by finiteness of (13), dominated convergence theorem applies and we get the result.  $\square$

The pulse (8) considered in [9] is a special case for Proposition 2 where the process  $Z$  is a zero-mean martingale. In fact, we have  $\mathbb{E}|Z| < \infty$  with the particular pulse (8) which has the shape of an isosceles triangle. In this case, the effect starts at time  $s$  as the buy or sell order is first given. It increases linearly as the amount traded increases, reaches its maximum value at time  $s + \frac{u}{2}$  as the highest effect is reached and starts decreasing from that point on until it vanishes at time  $s + u$  and brings the price level back to the original, locally. Such functions  $f$  represent the effect of an order which takes place over a period, and then vanishes after a while. This scenario is a milder and time limited version of the Poisson shot-noise of [28] where a shock in the market changes the price through a jump, but then it tends back to its initial level by an exponential decay of the first effect. There is no need to get back the effect as in the pulses of [9] for the limit theorems. The effect function may leave the price in a level different from the one it found at an arrival, in which case we consider the centered process.

We can show the connection of the two measures in propositions 1 and 2 in the next theorem. We introduce a scaling factor  $n \in \mathbb{Z}_+$  which will be taken to infinity in the limit. An integral with respect to a probability measure  $\nu(du)$  with regularly varying tail converges to an integral with respect to the measure  $u^{-\delta-1}du$  in the limit. We first need the following lemma.

**Lemma 1.** *Suppose  $f$  is a Lipschitz continuous function on  $\mathbb{R}$  with  $f(x) = 0$  for all  $x \leq 0$  and  $f(x) = f(1)$  for all  $x \geq 1$ . Then,*

$$\int_{-\infty}^{\infty} \int_0^{\infty} u^{1+\kappa} \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right]^{1+\kappa} u^{-\delta-1} du ds < \infty \quad (14)$$

for  $1 < \delta < 3$  and  $\kappa > 0$  such that  $1 + \kappa > \delta$ .

**Proof:** Let us call the integral (14) as  $I$ . In view of the assumptions on  $f$ , we have

$$\begin{aligned} |I| \leq M^{1+\kappa} & \left\{ \int_{-\infty}^0 \int_{-s}^{t-s} (u+s)^{1+\kappa} u^{-\delta-1} du ds + \int_0^t \int_0^{t-s} u^{1+\kappa} u^{-\delta-1} du ds \right. \\ & \left. + \int_0^t \int_{t-s}^{\infty} (t-s)^{1+\kappa} u^{-\delta-1} du ds + \int_{-\infty}^0 \int_{t-s}^{\infty} t^{1+\kappa} u^{-\delta-1} du ds \right\} \end{aligned}$$

which can be shown along the same lines as in the proof of Proposition 1. Evaluating the above integrals, we find that

$$|I| \leq M^{1+\kappa} t^{2+\kappa-\delta} \left[ \frac{1}{(2+\kappa)(2+\kappa-\delta)} + \frac{1}{(2+\kappa)\delta} + \frac{1}{(2+\kappa-\delta)(1+\kappa-\delta)} + \frac{1}{\delta(2+\kappa-\delta)} + \frac{1}{\delta(\delta-1)} \right]$$

□

**Theorem 1.** Suppose that  $\nu$  is a probability measure satisfying  $\int u \nu(du) < \infty$  and has a regularly varying tail as given in (3), the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz continuous with  $f(x) = 0$  for all  $x \leq 0$ ,  $f(x) = f(1)$  for all  $x \geq 1$  and is also differentiable with  $f'$  satisfying a Lipschitz condition a.e., and  $\mathbb{E}|R|^{1+\kappa} < \infty$  for some  $0 < \kappa \leq 1$  with  $1 + \kappa > \delta > 1$ . Let

$$\mu_n(ds, du, dr) = \frac{n^\delta}{h(n)} \lambda ds \nu_n(du) \gamma(dr)$$

where  $\nu_n(du) = \nu(du)$  and

$$Z_n(t) = \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} r u \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] N_n(ds, du, dr).$$

Then,  $\{Z_n(t) - \mathbb{E}Z_n(t), t \geq 0\}$  converges in law to the process

$$\left\{ \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} r u \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] \tilde{N}'(ds, du, dr), t \geq 0 \right\}$$

as  $n \rightarrow \infty$ , where  $\tilde{N}' = N' - \mu'$  for a Poisson random measure  $N'$  with mean measure  $\mu'(ds, du, dr) = \lambda u^{-\delta-1} ds du \gamma(dr)$ .

**Proof:** For the convergence of finite dimensional distributions of  $\{Z_n(t) - \mathbb{E}Z_n(t), t \geq 0\}$ , consider the characteristic function  $\mathbb{E} \exp i \sum_{k=1}^m \xi_k [Z_n(t_k) - \mathbb{E}Z_n(t_k)]$  for  $\xi_k \in \mathbb{R}$ ,  $t_k \geq 0$  and  $m \in \mathbb{N}$ . It is given by

$$\exp \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} \left\{ e^{i \sum_{k=1}^m \xi_k r u \left[ f\left(\frac{t_k-s}{u}\right) - f\left(\frac{-s}{u}\right) \right]} - 1 - i \sum_{k=1}^m \xi_k r u \left[ f\left(\frac{t_k-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] \right\} \frac{n^\delta}{h(n)} \lambda ds \nu_n(du) \gamma(dr). \quad (15)$$

We first show that the exponent in (15) is bounded and then use bounded convergence theorem to take the limit. This theorem is a generalization of [25, Thm.1] with the

general effect function  $f$ . Although we follow the same approach as in [25, Thm.1], there are more terms to bound in our case. Let

$$g(s, u, r) = e^{i \sum_{k=1}^m \xi_k r u \left[ f\left(\frac{t_k - s}{u}\right) - f\left(\frac{-s}{u}\right) \right]} - 1 - i \sum_{k=1}^m \xi_k r u \left[ f\left(\frac{t_k - s}{u}\right) - f\left(\frac{-s}{u}\right) \right]. \quad (16)$$

Using the random variable  $U$ , we denote the left hand side of (3) as  $\mathbb{P}\{U \geq u\}$  below.

By integration by parts, the exponent in (15) is equal to

$$\int \int \int \partial_u g(s, u, r) \mathbb{P}\{U > nu\} \frac{n^\delta}{h(n)} \lambda ds du \gamma(dr) \quad (17)$$

where  $\partial_u$  is  $\partial/\partial u$  and the hypothesis that  $\nu_n(du) = \nu(d(nu))$  is used.

**a)** Bound for the integrand of (17) for large values of  $u$ :

In view of Potter bounds [6], for  $\epsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that

$$\frac{\mathbb{P}\{U > nu\}}{\mathbb{P}\{U > n\}} \leq 2u^{-\delta} \max(u^{-\epsilon}, u^\epsilon)$$

for all  $n \geq n_0$  and  $nu \geq n_0$ , that is,  $u \geq n_0/n$ . Since  $\lim_{n \rightarrow \infty} \mathbb{P}\{U > n\} n^\delta / h(n) = C$  for some  $C > 0$ , we have  $\mathbb{P}\{U > n\} n^\delta / h(n) \leq (C + \epsilon)$  for all  $n \geq n'_0$  for some  $n'_0 \in \mathbb{N}$ . Note that  $C = 1/\delta$  by (3). Assume  $n'_0 \leq n_0$  for simplicity of notation. Therefore, we get

$$\mathbb{P}\{U > nu\} \frac{n^\delta}{h(n)} \leq 2u^{-\delta} \max(u^{-\epsilon}, u^\epsilon) (C + \epsilon) \quad (18)$$

for all  $n \geq n_0$  and  $u \geq n_0/n$ .

In (17), we have

$$\partial_u g(s, u, r) = i \left[ e^{i \sum_{k=1}^m \xi_k r u \left[ f\left(\frac{t_k - s}{u}\right) - f\left(\frac{-s}{u}\right) \right]} - 1 \right] \partial_u S(s, u, r) \quad (19)$$

where

$$\begin{aligned} \partial_u S(s, u, r) &:= \frac{\partial}{\partial u} \sum_{k=1}^m \xi_k r u \left[ f\left(\frac{t_k - s}{u}\right) - f\left(\frac{-s}{u}\right) \right] \\ &= \sum_k \xi_k r \left[ f\left(\frac{t_k - s}{u}\right) - f\left(\frac{-s}{u}\right) \right] \\ &\quad + \sum_k \xi_k r \left[ -f'\left(\frac{t_k - s}{u}\right) \frac{t_k - s}{u} + f'\left(\frac{-s}{u}\right) \frac{-s}{u} \right] \end{aligned}$$

Now, we can bound  $|\partial_u S|$  using the Lipschitz property of  $f$  and  $f'$  on different regions for  $u$  and  $s$ . Let  $M > 0$  and  $M' > 0$  stand for the Lipschitz constants of  $f$  and  $f'$ ,

respectively, or their upper bound, whichever is larger. Let us assume  $M > M'$  for simplicity of notation.

i)  $s < 0$  and  $0 < s + u < t_k$

Since  $(t_k - s)/u > 1$  and  $-s/u < 0$ , we have

$$\left| f\left(\frac{t_k - s}{u}\right) - f\left(\frac{-s}{u}\right) \right| = \left| f(1) - f\left(\frac{-s}{u}\right) \right| \leq M \left| 1 + \frac{s}{u} \right|$$

and

$$\left| -f'\left(\frac{t_k - s}{u}\right) \frac{t_k - s}{u} + f'\left(\frac{-s}{u}\right) \frac{-s}{u} \right| = \left| 0 - f'\left(\frac{-s}{u}\right) \frac{-s}{u} \right| \leq M' \left| \frac{s}{u} \right| \leq M \left| \frac{s}{u} \right|$$

due to the form of  $f$  and Lipschitz assumptions. Therefore, we get

$$|\partial_u S(s, u, r)| \leq \left( M \left| 1 + \frac{s}{u} \right| + M \left| \frac{s}{u} \right| \right) \sum_k \xi_k |r| = M \sum_k |\xi_k| |r|$$

since  $|1 + s/u| < 1$  and  $|s/u| < 1$  in this region.

ii)  $s > 0$  and  $s + u < t_k$

In this region,  $f'$  vanishes and  $f(-s/u) = 0$ . Therefore, we have

$$|\partial_u S(s, u, r)| = |f(1)| \leq M \sum_k |\xi_k| |r|.$$

iii)  $0 < s < t_k$  and  $t_k < s + u$

In this region,  $f(-s/u) = f'(-s/u) = 0$  and we get

$$|\partial_u S(s, u, r)| \leq 2M \sum_k |\xi_k| \left| \frac{t_k - s}{u} \right| |r|.$$

iv)  $s < 0$  and  $t_k < s + u$

We have

$$\left| f\left(\frac{t_k - s}{u}\right) - f\left(\frac{-s}{u}\right) \right| \leq M \left| \frac{t_k}{u} \right|$$

and

$$\left| f'\left(\frac{t_k - s}{u}\right) \frac{t_k - s}{u} - f'\left(\frac{-s}{u}\right) \frac{-s}{u} \right| \leq M' \left| \frac{t_k}{u} \right| + M' \left| \frac{st_k}{u^2} \right|.$$

The corresponding bound on  $|\partial_u S(s, u, r)|$  follows.

Now, we can bound the remaining terms in (19) by

$$2^{1-\kappa} \sum_j |\xi_j|^\kappa |r|^\kappa u^\kappa \left| f\left(\frac{t_j - s}{u}\right) - f\left(\frac{-s}{u}\right) \right|^\kappa \wedge 2 \quad (20)$$

using the inequalities  $|e^{ix} - 1| \leq 2^{1-\kappa}|x|^\kappa$  and  $\left(\sum_j |x_j|\right)^\kappa \leq \sum_j |x_j|^\kappa$ ,  $0 < \kappa \leq 1$  [25], and the fact that  $|e^{ix} - 1| \leq 2$ . The index  $k$  is replaced by  $j$  in order to distinguish the cross products of sums below. We further note that

$$\left| f\left(\frac{t_j - s}{u}\right) - f\left(\frac{-s}{u}\right) \right|^\kappa \leq (2M)^\kappa \leq 2^\kappa M \quad (21)$$

since  $f$  is bounded and  $M^\kappa \leq M$ , assuming  $M > 1$  for simplicity of notation. Putting all terms together by (18), (20), (21) and i)-iv), we find that (17) is bounded as

$$\begin{aligned} & \int \int_{n_0/n}^\infty \int |\partial_u g(s, u, r)| \mathbb{P}\{U > nu\} \frac{n^\delta}{h(n)} \lambda ds du \gamma(dr) \\ & \leq 4M(C + \epsilon) \sum_k |\xi_k| \int \int_0^\infty \int |r| B(s, u, t_k) \max(u^{-\epsilon}, u^\epsilon) u^{-\delta} \lambda ds du \gamma(dr) \end{aligned} \quad (22)$$

where

$$\begin{aligned} B(s, u, t_k) &= \left( 1 \wedge \sum_j M |\xi_j|^\kappa |r|^\kappa u^\kappa 1_{\{s \leq t_j\}} \right) \\ &\cdot \left[ 1_{R_{1,k}} + 1_{R_{2,k}} + 2 \frac{t_k - s}{u} 1_{R_{3,k}} + \left( 2 \frac{t_k}{u} + \frac{|s|t_k}{u^2} \right) 1_{R_{4,k}} \right] \end{aligned} \quad (23)$$

and  $R_{1,k}, \dots, R_{4,k}$  denote the regions in i)-iv). Since  $[1 \wedge \sum_j M |\xi_j|^\kappa |r|^\kappa u^\kappa 1_{\{s \leq t_j\}}] \leq 1$ , we can write

$$\begin{aligned} B(s, u, t_k) &\leq 1_{R_{1,k}} + 2 \frac{t_k - s}{u} 1_{R_{3,k}} + \left( 2 \frac{t_k}{u} + \frac{|s|t_k}{u^2} \right) 1_{R_{4,k}} \\ &\quad + \left( 1 \wedge \sum_j M |\xi_j|^\kappa |r|^\kappa u^\kappa 1_{\{s \leq t_j\}} \right) 1_{R_{2,k}} \end{aligned} \quad (24)$$

We keep the extra bounding term for  $R_{2,k}$ , as the integration in this region is more delicate. For fixed  $k \in \{1, \dots, n\}$ ,  $R_{1,k}, \dots, R_{4,k}$  are depicted in Fig.2. If we choose  $\epsilon > 0$  such that

$$1 < \delta - \epsilon < \delta < \delta + \epsilon < 1 + \kappa, \quad (25)$$

then the right hand side of (22) is finite as shown in Appendix A.

**b) Bound for the integrand of (17) for small values of  $u$ :**

We now consider  $u \leq n_0/n \leq 1$  as  $n \geq n_0$ . We use Markov inequality for  $\mathbb{P}\{U \geq nu\}$  as in [25], together with the bounds (20), (21) and i)-iv) in our case. We have

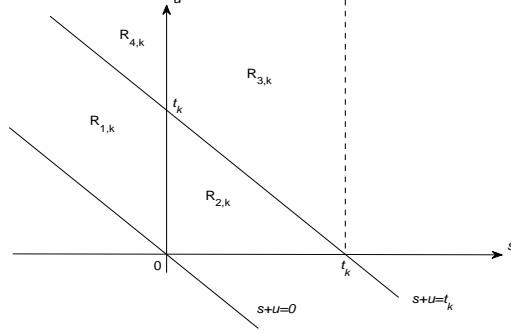


Figure 2: Subregions considered for  $(s, u)$

$\mathbb{P}\{U \geq nu\} \leq \mathbb{E}U/(nu)$  by Markov's inequality. Therefore, we get

$$|\partial_u g(s, u, r)| \mathbb{P}\{U > nu\} \frac{n^\delta}{h(n)} \leq 2M \frac{\mathbb{E}U}{u} \frac{n^{\delta-1}}{h(n)} |r| \sum_k |\xi_k| B(s, u, t_k) \quad (26)$$

From (24), we can write

$$B(s, u, t_k) \leq 1_{R_{1,k}} + 2 \frac{t_k - s}{u} 1_{R_{3,k}} + \left( 2 \frac{t_k}{u} + \frac{|s|t_k}{u^2} \right) 1_{R_{4,k}} + 1_{R_{2,k}} \sum_j M |\xi_j|^\kappa |r|^\kappa u^\kappa 1_{\{s \leq t_j\}}$$

considering that  $u$  is bounded as  $u \leq 1$ . Now, we have

$$1 = u^{\delta+\epsilon-1} u^{1-\delta-\epsilon} \leq n_0^{\delta+\epsilon-1} n^{-\epsilon} n^{1-\delta} u^{1-\delta-\epsilon} \leq n_0^{\delta+\epsilon-1} h(n) n^{1-\delta} u^{1-\delta-\epsilon} \quad (27)$$

since  $u \leq n_0/n$  and  $n^{-\epsilon} \leq h(n)$  for the slowly varying function  $h$  when  $n$  is sufficiently large [25]. Using (27) to increase the righthand side of (26) and in view of (24), we get

$$|\partial_u g(s, u, r)| \mathbb{P}\{U > nu\} \frac{n^\delta}{h(n)} \leq 2M \mathbb{E}U n_0^{\delta+\epsilon-1} |r| u^{-\delta-\epsilon} \sum_k |\xi_k| \cdot \left[ 1_{R_{1,k}} + 2 \frac{t_k - s}{u} 1_{R_{3,k}} + \left( 2 \frac{t_k}{u} + \frac{|s|t_k}{u^2} \right) 1_{R_{4,k}} + 1_{R_{2,k}} \sum_j M |\xi_j|^\kappa |r|^\kappa u^\kappa 1_{\{s \leq t_j\}} \right]$$

which is integrable over  $0 < u < 1$ , as in part a), in view of the computations in Appendix A.

As a result of a) and b), the integrand in (17) is bounded by an integrable function. Therefore, we can use dominated convergence theorem to find that

$$\lim_{n \rightarrow \infty} \mathbb{P}\{U > nu\} \frac{n^\delta}{h(n)} = \frac{u^{-\delta}}{\delta} \quad (28)$$

by (3) and (4), and then revert (17) by another integration by parts to get the limit of (15) as

$$\exp \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} g(s, u, r) \lambda u^{-\delta-1} ds du \gamma(dr)$$

where  $g$  is as in (16). It can be shown as in Proposition 2 that the above characteristic function and the corresponding process are well defined since  $|e^{ix} - 1 - ix|$  is bounded by  $|x|$ . Hence, we have shown the convergence of finite dimensional distributions.

To prove weak convergence in the Skorohod topology on  $D(0, \infty)$ , we first observe that

$$\begin{aligned} \mathbb{E}|Z_n(t) - \mathbb{E}Z_n(t)|^{1+\kappa} &\leq \\ 2\mathbb{E}|R|^{1+\kappa} \int_0^{\infty} \int_{-\infty}^t u^{1+\kappa} \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right]^{1+\kappa} \frac{n^\delta}{h(n)} \lambda ds \nu_n(du) \end{aligned} \quad (29)$$

by [25, Lemma 5]. By integration by parts and in view of Potter bounds as before, for  $\epsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that the part of the integral for  $u \geq n_0/n$  on the right hand side of (29) is bounded from above by

$$2(C + \epsilon) \int_{n_0/n}^{\infty} \int_{-\infty}^t \left| \frac{\partial}{\partial u} \left[ u^{1+\kappa} \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right]^{1+\kappa} \right] \right| \lambda u^{-\delta} \max\{u^{-\epsilon}, u^\epsilon\} ds du \quad (30)$$

We have

$$\begin{aligned} \frac{\partial}{\partial u} \left[ u^{1+\kappa} \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right]^{1+\kappa} \right] &= \\ (1+\kappa)u^\kappa \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right]^{1+\kappa} &+ \\ + (1+\kappa)u^\kappa \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right]^\kappa \left[ -f'\left(\frac{t-s}{u}\right) \frac{t-s}{u} - f'\left(\frac{-s}{u}\right) \frac{-s}{u} \right] \end{aligned}$$

an upper bound for the absolute value of which is given by

$$\begin{aligned} M^{1+\kappa}(1+\kappa) \{ [u^{-1}(u+s)^{1+\kappa} + u^{-1}s(u+s)^\kappa] 1_{R_1} + u^\kappa 1_{R_2} \\ + 2u^{-1}(t-s)^{1+\kappa} 1_{R_3} + u^{-1}t^{1+\kappa}(2+|s|) 1_{R_4} \} \end{aligned} \quad (31)$$

by Lipschitz assumptions on  $f$  and  $f'$ , where  $R_1, \dots, R_4$  are as in i) through iv) above, with  $t_k \equiv t$ . Substituting (31) in (30) and starting the lower limit for  $u$  from 0, we have an upper bound for the integral in (30) given by

$$\begin{aligned} & \int_{-\infty}^0 \int_{-s}^{t_k-s} [(u+s)^{1+\kappa} + s(u+s)^\kappa] u^{-\delta-1} \max(u^{-\epsilon}, u^\epsilon) du ds + \\ & \int_0^t \int_0^{t-s} u^{\kappa-\delta} \max(u^{-\epsilon}, u^\epsilon) du ds + 2 \int_0^{t_k} \int_{t-s}^\infty (t-s)^{1+\kappa} u^{-\delta-1} \max(u^{-\epsilon}, u^\epsilon) du ds \\ & + \int_{-\infty}^0 \int_{t-s}^\infty t^{1+\kappa} (2+|s|) u^{-\delta} \max(u^{-\epsilon}, u^\epsilon) du ds \end{aligned} \quad (32)$$

which is finite when we choose  $\epsilon$  as in (25). On the other hand, for  $0 < u < n_0/n$ , we use Markov's inequality as before to get

$$\mathbb{P}\{U > nu\} \frac{n^\delta}{h(n)} \leq \mathbb{E}U n_0^{\delta+\epsilon-1} u^{-\delta-\epsilon}$$

Then, the finiteness of the integrals in (32) is sufficient again for the integrability of a dominating function for  $0 < u < n_0/n < 1$  which complements (30). It follows from dominated convergence theorem that the limit of the right hand side of (29) exists. Therefore, possibly for  $n \geq n_1$  for some  $n_1 \in \mathbb{N}$ , the upper bound in (29) is further bounded by a multiple of its limit given by

$$C_1 \mathbb{E}|R|^{1+\kappa} \int_0^\infty \int_{-\infty}^t u^{1+\kappa} \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right]^{1+\kappa} \lambda u^{-\delta-1} du ds \quad (33)$$

for some  $C_1 > 2$ . In view of the proof of Lemma 1, the integral in (33) is bounded by a constant multiple of  $t^{2+\kappa-\delta}$  which clearly dominates  $\mathbb{E}|Z_n(t) - \mathbb{E}Z_n(t)|^{1+\kappa}$  in (29) for sufficiently large  $n$ . Since the increments of  $\{Z_n(t) - \mathbb{E}Z_n(t) : t \geq 0\}$  are stationary, this implies that

$$\begin{aligned} \mathbb{E}[|Z_n(t_2) - \mathbb{E}Z_n(t)|^{\frac{1+\kappa}{2}} |Z_n(t) - \mathbb{E}Z_n(t_1)|^{\frac{1+\kappa}{2}}] & \leq C_2 (t_2 - t)^{\frac{2+\kappa-\delta}{2}} (t - t_1)^{\frac{2+\kappa-\delta}{2}} \\ & \leq C_2 (t_2 - t_1)^{2+\kappa-\delta} \end{aligned} \quad (34)$$

for  $0 < t_1 < t < t_2$  and some  $C_2 > 0$ , by Cauchy-Schwarz inequality and the assumption that  $\delta < 1 + \kappa$ . This concludes the proof by [5, Thm.13.5 and Eqn.(13.14)] as  $2 + \kappa - \delta > 1$ .  $\square$

**Remark 3.** We consider the scaled measure  $\nu_n$  as a scaling of the parameters of  $\nu$ . For instance, if  $\nu$  was the Pareto distribution  $\nu(du) = \delta b^\delta u^{-\delta-1} du$  for  $u > b$ , with parameters  $\delta > 0$  and  $b > 0$ , we would have  $\nu_n(du) = \delta(b/n)^\delta u^{-\delta-1} du$  for  $u > b/n$ , which would amount to scaling the scale parameter  $b$  as  $b/n$ . This leads to the limiting infinite measure on  $\mathbb{R}_+$  as the cutoff parameter  $b$  decreases. Although similar interpretations are possible for other probability measures  $\nu$  as well, the scaling  $\nu_n(du) = \nu(d(nu))$  is essentially a time scaling. The random variable  $U$ , the duration of the sessions, has time interpretation. We look at the workload of individual pulses over shorter time periods by scaling  $U$  as  $U/n$ .

Note that the limiting process given in Theorem 1, specifically with  $f$  of (7), is obtained as a limit of sums of scaled renewal processes in [18] as shown in [19].

#### 4. Fractional Brownian Motion Limit

Fractional Brownian motion is a mean zero Gaussian process  $Z$  on  $\mathbb{R}_+$  with  $Z(0) = 0$  and covariance

$$\text{Cov}(Z(t_1), Z(t_2)) = \frac{\sigma^2}{2} (|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H}) \quad t_1, t_2 \geq 0$$

for  $t_1, t_2 \geq 0$ ,  $\sigma > 0$  and Hurst parameter  $0 < H < 1$  [33].

In this section, we scale the log-price process as follows to approximate a fractional Brownian motion in the limit. Let the rate  $R$  be scaled as  $R/n$  which can be interpreted as a decrease in the effect of an order in absolute value as  $n$  increases. This decrease may have arisen from an underlying decrease in the volume of the transaction, for example. On the other hand, we will let the arrival rate  $\lambda$  of orders increase with a factor which is a function of  $n$ . For each  $n \in \mathbb{Z}_+$ , let  $N_n$  denote the Poisson random measure with scaled mean measure  $\mu_n$  that involves the scaled arrival rate and possibly further scalings. In the following theorems, we prove convergence of  $Z_n - \mathbb{E}Z_n$  to fBm with a properly scaled measure  $\nu_n$  for a finite measure  $\nu$  as in (3) and with  $\nu$  as in (5). Compensated Poisson random measure is used when  $\mathbb{E}Z_n$  does not exist.

**Lemma 2.** *Let  $(x_n)$  be a strictly positive real sequence and  $c \in \mathbb{R}$ . If  $\lim(x_n) = \infty$ , then*

$$\lim_{n \rightarrow \infty} x_n^2 (e^{ic/x_n} - 1 - ic/x_n) = -\frac{c^2}{2}.$$

**Proof:** Consider the Taylor expansion [17, pg.184-186] of the function  $h(x) = e^{ic/x}$  around 0 on the disk  $D = \{x \in \mathbb{R} : |x| < 2\}$ . We have

$$x_n^2 \left| (e^{ic/x_n} - 1 - ic/x_n) + \frac{c^2}{2x_n^2} \right| = x_n^2 |R_2(x_n)|$$

where  $R_2$  is the remainder after 3 terms. For fixed  $r$  with  $1 < r < 2$ , we can express  $R_2$  as

$$R_2(x_n) = \frac{1}{2\pi i x_n^3} \int_{|x|=r} \frac{h(x)}{x^3(x - 1/x_n)} dx$$

By continuity of  $h$ , there exists  $M > 0$  such that  $|h(x)| \leq M$  for all  $x \in D$ , in particular for  $|x| = r$ . Clearly,  $|x - 1/x_n| \geq |x| - |1/x_n| = r - 1/x_n$  since  $x_n$  is positive. Then, we have the following bound for  $|R_2(x_n)|$ :

$$|R_2(x_n)| \leq \frac{Mr}{r - 1/x_n} \left( \frac{1/x_n}{r} \right)^3.$$

Taking the limit as  $n \rightarrow \infty$ , we obtain  $x_n^2 |R_2(x_n)| \rightarrow 0$ .  $\square$

**Theorem 2.** Suppose that  $\nu$  is a probability measure satisfying  $\int u \nu(du) < \infty$  and has a regularly varying tail as given in (3), the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz continuous with  $f(x) = 0$  for all  $x \leq 0$ ,  $f(x) = f(1)$  for all  $x \geq 1$  and is also differentiable with  $f'$  satisfying a Lipschitz condition a.e., and  $\mathbb{E}R^2 < \infty$ . Let

$$Z_n(t) = \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} \frac{r}{n} u \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] N_n(ds, du, dr)$$

and

$$\mu_n(ds, du, dr) = \frac{n^{2+\delta}}{h(n)} \lambda ds \nu_n(du) \gamma(dr)$$

where  $\nu_n(du) = \nu(d(nu))$  and  $1 < \delta < 2$ . Then, the process  $\{Z_n(t) - \mathbb{E}Z_n(t), t \geq 0\}$  converges in law to an fBm with variance parameter

$$\sigma^2 = \lambda \mathbb{E}R^2 \int_{-\infty}^{\infty} \int_0^{\infty} \left[ f\left(\frac{1-s}{u}\right) - f\left(\frac{-s}{u}\right) \right]^2 u^{1-\delta} du ds$$

as  $n \rightarrow \infty$ .

**Proof:** The characteristic function  $\mathbb{E} \exp i \sum_{k=1}^m \xi_k [Z_n(t_k) - \mathbb{E}Z_n(t_k)]$  for  $\xi_k \in \mathbb{R}$ ,  $t_k \geq 0$  and  $m \in \mathbb{N}$  is given by

$$\begin{aligned} & \exp \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} \left\{ e^{i \sum_{k=1}^m \xi_k \frac{r}{n} u \left[ f\left(\frac{t_k-s}{u}\right) - f\left(\frac{-s}{u}\right) \right]} - 1 \right. \\ & \quad \left. - i \sum_{k=1}^m \xi_k \frac{r}{n} u \left[ f\left(\frac{t_k-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] \right\} \frac{n^{2+\delta}}{h(n)} \lambda ds \nu_n(du) \gamma(dr). \quad (35) \end{aligned}$$

The same approach will be followed as in the proof of Theorem 1. By integration by parts, we find that the exponent of (35) is given by

$$\int \int \int \partial_u g(s, u, r/n) \mathbb{P}\{U > nu\} \frac{n^{2+\delta}}{h(n)} \lambda ds du \gamma(dr). \quad (36)$$

Using Potter bounds [6] and Lipschitz conditions on  $f$  and  $f'$ , we get an inequality similar to (22) for  $u \geq n_0/n$  given by

$$\begin{aligned} & \int \int_{n_0/n}^{\infty} \int |\partial_u g(s, u, r/n)| \mathbb{P}\{U > nu\} \frac{n^{2+\delta}}{h(n)} \lambda ds du \gamma(dr) \\ & \leq 4M^2(C + \epsilon) \sum_k |\xi_k| \int \int_0^{\infty} |r| \tilde{B}(s, u, t_k) \max(u^{-\epsilon}, u^{\epsilon}) u^{-\delta} \lambda ds du \gamma(dr) \end{aligned} \quad (37)$$

where  $\tilde{B}$  is similar to (23) but with  $\kappa = 1$  by hypothesis, and  $\epsilon > 0$  and  $n_0 \in \mathbb{N}$ . Precisely,

$$\begin{aligned} \tilde{B}(s, u, t_k) = & \left( 1 \wedge \sum_j M |\xi_j| |r| u 1_{\{s \leq t_j\}} \right) \\ & \cdot \left[ 1_{R_{1,k}} + 1_{R_{2,k}} + 2 \frac{t_k - s}{u} 1_{R_{3,k}} + \left( 2 \frac{t_k}{u} + \frac{|s| t_k}{u^2} \right) 1_{R_{4,k}} \right] \end{aligned}$$

If we choose  $\epsilon > 0$  such that

$$1 < \delta - \epsilon < \delta < \delta + \epsilon < 2,$$

then the right hand side of (37) is finite along the same lines of the proof of Theorem 1 with  $\kappa = 1$ . On the other hand, we can bound (36) for  $0 < u \leq 1$  similarly. Therefore, we can use dominated convergence theorem. We have

$$\lim_{n \rightarrow \infty} \mathbb{P}\{U > nu\} \frac{n^{\delta}}{h(n)} = \frac{u^{-\delta}}{\delta}$$

as in (28), and

$$\lim_{n \rightarrow \infty} n^2 \partial_u g(s, u, r/n) = \partial_u \lim_{n \rightarrow \infty} n^2 g(s, u, r/n)$$

as  $g$  is bounded, hence, uniformly continuous. Then, we get

$$\begin{aligned} & \lim_{n \rightarrow \infty} n^2 g(s, u, r/n) = \\ & -\frac{1}{2} \sum_{k=1}^m \sum_{j=1}^m \xi_j \xi_k r^2 \left[ f\left(\frac{t_j - s}{u}\right) - f\left(\frac{-s}{u}\right) \right] \left[ f\left(\frac{t_k - s}{u}\right) - f\left(\frac{-s}{u}\right) \right] u^2 \end{aligned} \quad (38)$$

by Lemma 2. We now revert (36) after the limits above, by another integration by parts, and get the limit of (35) as

$$\exp \left\{ - \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \sum_{k=1}^m \sum_{j=1}^m \xi_j \xi_k r^2 \left[ f \left( \frac{t_j - s}{u} \right) - f \left( \frac{-s}{u} \right) \right] \left[ f \left( \frac{t_k - s}{u} \right) - f \left( \frac{-s}{u} \right) \right] u^2 \lambda ds u^{-\delta-1} du \gamma(dr) \right\}$$

which is the characteristic function of  $\sum_{k=1}^m \xi_k Z(t_k)$ , where  $Z = (Z(t_1), \dots, Z(t_m))$  is a Gaussian vector with zero mean and covariance

$$\lambda \mathbb{E} R^2 \int_0^{\infty} \int_{-\infty}^{\infty} \left[ f \left( \frac{t_j - s}{u} \right) - f \left( \frac{-s}{u} \right) \right] \left[ f \left( \frac{t_k - s}{u} \right) - f \left( \frac{-s}{u} \right) \right] u^2 ds u^{-\delta-1} du. \quad (39)$$

When (39) is evaluated at  $t_j = t_k = 1$ , the variance coefficient is found to be

$$\sigma^2 := \text{Var}(Z(1)) = \lambda \mathbb{E} R^2 \int_0^{\infty} \int_{-\infty}^{\infty} \left[ f \left( \frac{1-s}{u} \right) - f \left( \frac{-s}{u} \right) \right]^2 ds u^{1-\delta} du$$

which is finite by Lemma 1 with  $\kappa = 1$ . Using the identity  $2ab = -(a-b)^2 + a^2 + b^2$  for  $a, b \in \mathbb{R}$  and making several change of variables, we find that the covariance of  $Z$  in (39) is given by

$$\text{Cov}(Z(t_j), Z(t_k)) = \frac{\sigma^2}{2} (t_j^{2H} + t_k^{2H} - |t_j - t_k|^{2H})$$

for  $t_j, t_k \geq 0$  with  $H = (3 - \delta)/2$ . By definition,  $Z$  has the characteristic function of an fBm.

Convergence in the Skorohod topology on  $D(0, \infty)$  follows along the same lines of proof of Theorem 1. In this case, we have

$$\mathbb{E} |Z_n(t) - \mathbb{E} Z_n(t)|^2 = \mathbb{E} R^2 \int_0^{\infty} \int_{-\infty}^t u^2 \left[ f \left( \frac{t-s}{u} \right) - f \left( \frac{-s}{u} \right) \right]^2 \lambda ds \nu_n(du)$$

and (34) holds with  $\kappa = 1$ . □

As an example, the continuous flow rate model studied in [25] is given by

$$Z(t) = \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} [(t-s)^+ \wedge u - (-s)^+ \wedge u] r N(ds, du, dr) \quad (40)$$

with  $f$  replaced by the special form (7) in (9). In [25, Thm.2], the limit is studied when the speed of time increases in proportion to the intensity of Poisson arrivals. To

balance the increasing trading intensity  $\lambda_n$ , time is speeded up by a factor  $n$  and the size is normalized by a factor  $\lambda_n^{1/2} n^{(3-\delta)/2}$  provided that  $\lambda_n/n^{\delta-1} \rightarrow \infty$ . We can let  $\lambda_n = n^{\varepsilon+\delta-1}$  with  $\varepsilon > 0$ . Taking  $\varepsilon = 2$ , we show the equivalence of the scaling of [25, Thm.2] to the scaling in Theorem 2. Note that  $\lambda_n = n^{1+\delta}$ . The scaled and centered process has the form

$$\begin{aligned} & \frac{Z(nt) - \mathbb{E}Z(nt)}{\lambda_n^{1/2} n^{(3-\delta)/2}} \\ &= \frac{1}{n^2} \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} ru \left[ f\left(\frac{nt-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] \tilde{N}_n(ds, du, dr) \end{aligned} \quad (41)$$

where we have written an effect function  $f$  in general. Then, we can make change of variables  $s \rightarrow ns$  and  $u \rightarrow nu$  to get

$$\begin{aligned} & \frac{Z(nt) - \mathbb{E}Z(nt)}{n^2} \\ &= \frac{1}{n^2} \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} r nu \left[ f\left(\frac{nt-ns}{nu}\right) - f\left(\frac{-ns}{nu}\right) \right] \tilde{N}_n(d(ns), d(nu), dr) \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} \frac{r}{n} u \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] \tilde{N}_n(d(ns), d(nu), dr) \end{aligned} \quad (42)$$

where the mean measure is

$$\begin{aligned} \mu_n(d(ns), d(nu), dr) &= \lambda_n(nds) \nu(d(nu)) \gamma(dr) \\ &= n^{2+\delta} ds \nu(d(nu)) \gamma(dr). \end{aligned} \quad (43)$$

In Theorem 2, we start with the scaled process (42) essentially. This can be observed by the fact that

$$N_n(d(ns), d(nu), dr) \stackrel{d}{=} N'_n(ds, du, dr)$$

for a Poisson random measure  $N'$  with mean measure

$$\mu'_n(ds, du, dr) = \mu_n(d(ns), d(nu), dr)$$

by definition of a Poisson random measure [24], [12, Def.V.2.2]. Equivalence of the scalings in Theorem 2 and [25, Thm.2] is in distributional sense. However, this is sufficient for equivalence as the convergence results are in distribution rather than almost sure sense. Therefore, we can apply Theorem 2 to obtain the limit as a fBm

with variance parameter

$$\sigma^2 = \mathbb{E} R^2 \int_0^\infty \int_{-\infty}^\infty [(1-s)^+ \wedge u - (-s)^+ \wedge u]^2 ds u^{-\delta-1} du = \frac{\mathbb{E} R^2}{(2-\delta)(3-\delta)}.$$

It is shown in [25] that the asymptotic behavior of the ratio  $\lambda_n/n^{\delta-1}$  determines the type of the limit process when time is speeded up by a factor  $n$ . For a choice of sequences  $\lambda_n$  and  $n$ , the random variable  $\sharp(\lambda_n, n)$  denotes the number of effects still active at time  $n$ . It measures the amount of very long pulses that are alive and how much they contribute to the total price. The expected value of the random variable  $\sharp(\lambda_n, n)$  is

$$\mathbb{E} \sharp(\lambda_n, n) \sim \frac{\lambda_n}{n^{\delta-1}}$$

for large  $n$ . The limit is considered in the cases where this value tends to a finite positive constant, to infinity, or to zero as  $\lambda_n$  and  $n$  go to infinity. We have already studied the case of finite constant in Theorem 1 and infinity in Theorem 2, the so-called intermediate and fast connection rates, respectively, in view of telecommunication applications. As shown above, our scalings do not involve time scaling. They can be physically understood as scalings of the parameters of the log-price process. The slow connection rate will be investigated similarly in terms of the model parameters in Theorem 4 in the next section.

The next theorem is a simpler version of Theorem 2 due to the form of the measure  $\nu$ . Note that (43) can be approximated as

$$\begin{aligned} \mu_n(d(ns), d(nu), dr) &\sim n^{2+\delta} ds n^{-\delta} u^{-\delta-1} du \gamma(dr) \\ &= n^2 u^{-\delta-1} ds du \gamma(dr) \end{aligned}$$

for large  $n$ . This scaling is used below with the simpler form of  $\nu$ . It can be interpreted as half way in taking the more involved limit of Theorem 2.

**Theorem 3.** *Let*

$$\tilde{Z}_n(t) = \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty \frac{r}{n} u \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] \tilde{N}_n(ds, du, dr)$$

where  $\tilde{N} = N - \mu$  and

$$\mu_n(ds, du, dr) = n^2 \lambda u^{-\delta-1} ds du \gamma(dr).$$

Suppose that  $\mathbb{E}R^2 < \infty$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a Lipschitz continuous function satisfying either of the following conditions

- i.  $f(x) = 0$  for all  $x \leq 0$  and  $f(x) = f(1)$  for all  $x \geq 1$ , or
- ii.  $f$  has a compact support.

Then, the process  $\{\tilde{Z}_n(t), t \geq 0\}$ , for  $1 < \delta < 3$ , converge in law to an fBm with variance parameter

$$\sigma^2 = \lambda \mathbb{E}R^2 \int_{-\infty}^{\infty} \int_0^{\infty} \left[ f\left(\frac{1-s}{u}\right) - f\left(\frac{-s}{u}\right) \right]^2 u^{1-\delta} du ds$$

as  $n \rightarrow \infty$ .

**Proof:** Although it can be found from the characteristic function of  $Z(t)$  in Proposition 2 that  $\mathbb{E}Z(t) = 0$  for all  $t \geq 0$  under assumption ii, we form  $\tilde{Z}$  as above since  $\mathbb{E}Z(t)$  may not exist with assumption i. For the convergence of finite dimensional distributions of  $\{\tilde{Z}_n(t), t \geq 0\}$ , consider the characteristic function  $\mathbb{E} \exp i \sum_{k=1}^m \xi_k \tilde{Z}_n(t_k)$  for  $\xi_k \in \mathbb{R}$ ,  $t_k \geq 0$  and  $m \in \mathbb{N}$ . It is given by

$$\exp \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} g(s, u, r/n) n^2 \lambda u^{-\delta-1} ds du \gamma(dr) \quad (44)$$

where  $g$  is given in (16). Note that the characteristic function exists since  $\tilde{Z}_n(t_k)$  are well defined in view of (14) which follows from Lemma 1 with  $\kappa = 1$  under assumption i, and by Proposition 2 under assumption ii. As  $n \rightarrow \infty$ , we will show that the above characteristic function converges to

$$\exp \left\{ - \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \sum_{k=1}^m \sum_{j=1}^m \xi_j \xi_k r^2 \left[ f\left(\frac{t_j - s}{u}\right) - f\left(\frac{-s}{u}\right) \right] \left[ f\left(\frac{t_k - s}{u}\right) - f\left(\frac{-s}{u}\right) \right] u^2 \lambda u^{-\delta-1} ds du \gamma(dr) \right\} \quad (45)$$

Due to the inequality  $|e^{ix} - 1 - ix| < \frac{1}{2}x^2$  for  $x \in \mathbb{R}$ , the integrand in (45) is an upper bound to

$$|g(s, u, r/n)| n^2$$

Therefore, dominated convergence theorem allows us to take the limit inside the integral in (44). That is, we must find

$$\lim_{n \rightarrow \infty} \left( e^{i \sum_{k=1}^m \xi_k \frac{r}{n} u \left[ f\left(\frac{t_k - s}{u}\right) - f\left(\frac{-s}{u}\right) \right]} - 1 - i \sum_{k=1}^m \xi_k \frac{r}{n} u \left[ f\left(\frac{t_k - s}{u}\right) - f\left(\frac{-s}{u}\right) \right] \right) n^2$$

which is now equal to

$$-\frac{1}{2} \sum_{k=1}^m \sum_{j=1}^m \xi_j \xi_k r^2 \left[ f\left(\frac{t_j - s}{u}\right) - f\left(\frac{-s}{u}\right) \right] \left[ f\left(\frac{t_k - s}{u}\right) - f\left(\frac{-s}{u}\right) \right] u^2 \quad (46)$$

by Lemma 2. This shows that (44) converges to (45) as  $n \rightarrow \infty$  by the continuity of the exponential function. The variance is evaluated in the proof of Theorem 2.

To complete the proof, we need to show convergence in  $D(0, \infty)$  with Skorohod topology. This is straight forward since the variance of  $\tilde{Z}_n(t)$  is already free of  $n$  and is bounded by a constant multiple of  $t^{3-\delta}$  by the proof of Lemma 1.  $\square$

**Remark 4.** Theorem 3 with condition ii. is Theorem 3.1 of [9] where it is noted that a fractional Brownian motion with  $H > 1/2$  can be approximated if the pulse is continuous and has a compact support. Condition i. above considers an effect function which is continuous, but with no compact support as an alternative.

## 5. Lévy Process Limit

A process with stationary and independent increments is called a Lévy process [4, 37]. The results of this section concerns a particular class of Lévy processes, namely stable Lévy motion [33]. Let  $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$ , and let  $\delta \in (1, 2)$ , and  $\beta \in [-1, 1]$  be the index of stability and skewness parameter, respectively. Then, a  $\delta$ -stable Lévy motion  $L$  with mean 0 can be defined through its characteristic function

$$\mathbb{E} e^{i\xi L(t)} = \exp\{-t \sigma^\delta |\xi|^\delta [1 - i\beta(\text{sign } \xi) \tan(\pi\delta/2)]\}$$

for  $\xi \in \mathbb{R}$ , where  $\sigma \geq 0$  is a scale parameter.

In this section, we prove that the limiting process is a  $\delta$ -stable Lévy motion under different scalings of the price process. Theorem 4 considers a probability measure  $\nu$  and Theorem 5 starts with its limiting form. For simplicity of notation, we take  $f(1) = 1$  in the effect function.

**Lemma 3.** *Let  $N$  be a Poisson random measure with mean measure*

$$\mu = \lambda ds u^{-\delta-1} du \gamma(dr)$$

*and  $\tilde{N} = N - \mu$ . Then,*

$$\int_{-\infty}^{\infty} \int_0^{\infty} \int_0^t r u \tilde{N}(ds, du, dr) \stackrel{d}{=} (\lambda C_1)^{1/\delta} L_1(t) + (\lambda C_2)^{1/\delta} L_2(t)$$

where  $C_1 = \int_0^\infty r^\delta \gamma(dr)$ ,  $C_2 = \int_{-\infty}^0 |r|^\delta \gamma(dr)$ , and  $L_1$  and  $L_2$  are independent  $\delta$ -stable Lévy motions with mean 0, skewness intensity  $\beta$  equal to 1 and  $-1$ , respectively, and scale parameter

$$\sigma = \left[ -\frac{2\Gamma(2-\delta)}{\delta(\delta-1)} \cos \frac{\pi\delta}{2} \right]^{1/\delta}.$$

**Proof:** Putting  $u' = |r|u$ , we get

$$\begin{aligned} & \int_{-\infty}^\infty \int_0^\infty \int_0^t r u \tilde{N}(ds, du, dr) \\ &= \int_0^\infty \int_0^\infty \int_0^t u' [N(ds, d(u'/r), dr) - \lambda ds u'^{-\delta-1} r^\delta du' \gamma(dr)] \\ &\quad - \int_{-\infty}^0 \int_0^\infty \int_0^t u' [N(ds, d(u'/|r|), dr) - \lambda ds u'^{-\delta-1} |r|^\delta du' \gamma(dr)] \\ &= \int_0^\infty \int_0^t u' [N'_1(ds, du') - C_1 \lambda ds u'^{-\delta-1} du'] \\ &\quad - \int_0^\infty \int_0^t u' [N'_2(ds, du') - C_2 \lambda ds u'^{-\delta-1} du'] \end{aligned} \tag{47}$$

where we have put

$$\begin{aligned} N'_1(ds, du') &:= \int_{\{r:r \in (0, \infty)\}} N(ds, d(u'/r), dr) \\ N'_2(ds, du') &:= \int_{\{r:r \in (-\infty, 0)\}} N(ds, d(u'/|r|), dr) \end{aligned}$$

and

$$C_1 := \int_0^\infty r^\delta \gamma(dr), \quad C_2 := \int_{-\infty}^0 |r|^\delta \gamma(dr),$$

It is easy to verify that both  $N'_1$  and  $N'_2$  are Poisson random measures with means  $C_i \lambda ds u'^{-\delta-1} du'$ ,  $i = 1, 2$ , respectively. What is more, they are independent as their domains are disjoint. Making another change of variable  $u = u'/(\lambda C_i)^{1/\delta}$  for  $i = 1, 2$  in respective integrals in (47), we get

$$\begin{aligned} & \int_{-\infty}^\infty \int_0^\infty \int_0^t r u \tilde{N}(ds, du, dr) \\ &= (\lambda C_1)^{1/\delta} \int_0^\infty \int_0^t u [N'_1(ds, d((\lambda C_1)^{1/\delta} u)) - ds u^{-\delta-1} du] \\ &\quad - (\lambda C_2)^{1/\delta} \int_0^\infty \int_0^t u [N'_2(ds, d((\lambda C_2)^{1/\delta} u)) - ds u^{-\delta-1} du] \\ &=: (\lambda C_1)^{1/\delta} L'_1(t) - (\lambda C_2)^{1/\delta} L'_2(t) \end{aligned} \tag{48}$$

where  $L'_1$  and  $L'_2$  are independent  $\delta$ -stable Lévy motions with skewness parameter  $\beta = 1$  and scale parameter

$$\sigma = \left[ -\frac{2\Gamma(2-\delta)}{\delta(\delta-1)} \cos \frac{\pi\delta}{2} \right]^{1/\delta}$$

for  $\xi \in \mathbb{R}$  [33, pg.s 5,156,350]. Now, we have

$$-L'_2 \stackrel{d}{=} L_2 := \int_{-\infty}^0 \int_0^t u [N'_2(ds, d(-(\lambda C_2)^{1/\delta} u)) - ds |u|^{-\delta-1} du]$$

where  $L_2$  is also a  $\delta$ -stable Lévy motion since  $N''_2(ds, du) := N'_2(ds, d(-(\lambda C_2)^{1/\delta} u))$  is a Poisson random measure on  $\mathbb{R}_+ \times \mathbb{R}_-$  with mean measure  $ds |u|^{-\delta-1} du$ , but skewness parameter  $\beta = -1$  [33, pg.5]. We can take  $L_1 = L'_1$  and the result follows.  $\square$

**Theorem 4.** Suppose that  $\nu$  is a probability measure satisfying  $\int u \nu(du) < \infty$  and has a regularly varying tail as given in (3), the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz continuous with  $f(x) = 0$  for all  $x \leq 0$ ,  $f(x) = f(1) = 1$  for all  $x \geq 1$  and is also differentiable with  $f'$  satisfying a Lipschitz condition a.e., and  $\mathbb{E}|R|^{1+\kappa} < \infty$  for some  $0 < \kappa \leq 1$  such that  $1 + \kappa > \delta > 1$ . Let

$$Z_n(t) = \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} n^{1-\alpha/\delta} r u \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] N_n(ds, du, dr)$$

and

$$\mu_n(ds, du, dr) = \frac{n^\alpha}{h(n^{\alpha/\delta})} \lambda ds \nu_n(du) \gamma(dr)$$

where  $\nu_n(du) = \nu(nu)$  and  $0 < \alpha < \delta$ . Then, the process  $\{Z_n(t) - \mathbb{E}Z_n(t), t \geq 0\}$ , for  $1 < \delta < 2$ , converges in law to

$$(\lambda \mathbb{E}R^\delta 1_{\{R>0\}})^{1/\delta} L_1(t) + (\lambda \mathbb{E}|R|^\delta 1_{\{R<0\}})^{1/\delta} L_2(t)$$

as  $n \rightarrow \infty$ , where  $L_1$  and  $L_2$  are independent  $\delta$ -stable Lévy motions with mean 0, and skewness intensity 1 and  $-1$ , respectively.

**Proof:** The characteristic function of the scaled process is given by

$$\exp \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} g(s, u, n^{1-\alpha/\delta} r) \frac{n^\alpha}{h(n^{\alpha/\delta})} \lambda ds \nu_n(du) \gamma(dr)$$

where  $g$  is as in (16). By integration by parts, we get

$$\exp \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} \partial_u g(s, u, n^{1-\alpha/\delta} r) \mathbb{P}\{U > nu\} \frac{n^\alpha}{h(n^{\alpha/\delta})} \lambda ds du \gamma(dr) \quad (49)$$

since  $\nu_n(du) = \nu(d(nu))$ . Making a change of variable  $u$  to  $u/n^{1-\alpha/\delta}$ , the logarithm of (49) is equal to

$$\int \int \int \frac{1}{n^{1-\alpha/\delta}} \partial_u g(s, u/n^{1-\alpha/\delta}, n^{1-\alpha/\delta}r) \mathbb{P}\{U > n^{\alpha/\delta}u\} \frac{n^\alpha}{h(n^{\alpha/\delta})} \lambda ds du \gamma(dr). \quad (50)$$

Using Potter bounds and Lipschitz conditions on  $f$  and  $f'$ , we get an inequality similar to (22). We can bound  $\mathbb{P}\{U > n^{\alpha/\delta}u\} n^\alpha/h(n^{\alpha/\delta})$  as in (18) and consider  $|\partial_u g(s, u/n^{1-\alpha/\delta}, n^{1-\alpha/\delta}r)|/n^{1-\alpha/\delta}$  separately. As a result, for fixed  $\epsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that for all  $n$  with  $n^{\alpha/\delta} \geq n_0$ , we have the following upper bound for the absolute value of (50) when evaluated over  $u \geq n_0/n^{\alpha/\delta}$

$$4M(C + \epsilon) \sum_k |\xi_k| \int \int_0^\infty \int |r| B'(s, u, t_k, n) \max(u^{-\epsilon}, u^\epsilon) u^{-\delta} \lambda ds du \gamma(dr)$$

where  $B'$  is analogous to (23) satisfying

$$\begin{aligned} B'(s, u, t_k, n) &\leq 1_{R_{1,k,n}} + \left( 1 \wedge \sum_j M |\xi_j|^\kappa |r|^\kappa u^\kappa 1_{\{s \leq t_j\}} \right) \cdot 1_{R_{2,k,n}} \\ &+ 2 \frac{t_k - s}{u/n^{1-\alpha/\delta}} 1_{R_{3,k,n}} + \left( 2 \frac{t_k}{u/n^{1-\alpha/\delta}} + \frac{|s|t_k}{u^2/n^{2(1-\alpha/\delta)}} \right) 1_{R_{4,k,n}} \end{aligned} \quad (51)$$

and  $R_{1,k,n}, \dots, R_{4,k,n}$  are analogous to  $R_{1,k}, \dots, R_{4,k}$  with  $u$  replaced by  $u/n^{1-\alpha/\delta}$ . The right hand side of (51) is integrable when  $\epsilon$  is chosen as in (25) as shown in Appendix B.

For  $u < n_0/n^{\alpha/\delta}$ , we can find a dominating function for the integrand in (50) using Markov's inequality. As in the proof of Theorem 1, we have

$$\begin{aligned} \frac{1}{n^{1-\alpha/\delta}} |\partial_u g(s, u/n^{1-\alpha/\delta}, n^{1-\alpha/\delta}r)| \mathbb{P}\{U > n^{\alpha/\delta}u\} \frac{n^\alpha}{h(n^{\alpha/\delta})} \\ \leq 2M \frac{\mathbb{E}U}{u} \frac{n^{\alpha-\alpha/\delta}}{h(n^{\alpha/\delta})} |r| \sum_k |\xi_k| B'(s, u, t_k, n) \end{aligned} \quad (52)$$

where  $B'$  satisfies (51). Using (51) and using an inequality similar to (27) in view of the assumption  $u < n_0/n^{\alpha/\delta}$ , we can increase the righthand side of (52) as

$$\begin{aligned} 2M \mathbb{E}U n_0^{\delta+\epsilon-1} |r| u^{-\delta-\epsilon} \sum_k |\xi_k| [1_{R_{1,k,n}} + 1_{R_{2,k,n}} \sum_j M |\xi_j|^\kappa |r|^\kappa u^\kappa 1_{\{s \leq t_j\}} \\ + 2 \frac{t_k - s}{u/n^{1-\alpha/\delta}} 1_{R_{3,k,n}} + \left( 2 \frac{t_k}{u/n^{1-\alpha/\delta}} + \frac{|s|t_k}{u^2/n^{2(1-\alpha/\delta)}} \right) 1_{R_{4,k,n}}] \end{aligned} \quad (53)$$

which is integrable over  $0 < u < 1$  in view of Appendix B.

We can now use the dominated convergence theorem. Note that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n^{1-\alpha/\delta}} \partial_u g(s, u/n^{1-\alpha/\delta}, n^{1-\alpha/\delta} r) &= \lim_{n \rightarrow \infty} \partial_u [g(s, u/n^{1-\alpha/\delta}, n^{1-\alpha/\delta} r)] \\ &= \partial_u \lim_{n \rightarrow \infty} g(s, u/n^{1-\alpha/\delta}, n^{1-\alpha/\delta} r) \end{aligned} \quad (54)$$

where  $g(s, u/n^{1-\alpha/\delta}, n^{1-\alpha/\delta} r)$  is given by

$$e^{i \sum_{k=1}^m \xi_k r u \left[ f\left(\frac{t_k - s}{u/n^{1-\alpha/\delta}}\right) - f\left(\frac{-s}{u/n^{1-\alpha/\delta}}\right) \right]} - 1 - i \sum_{k=1}^m \xi_k r u \left[ f\left(\frac{t_k - s}{u/n^{1-\alpha/\delta}}\right) - f\left(\frac{-s}{u/n^{1-\alpha/\delta}}\right) \right] \quad (55)$$

and we have

$$\lim_{n \rightarrow \infty} \left[ f\left(\frac{t_k - s}{u/n^{1-\alpha/\delta}}\right) - f\left(\frac{-s}{u/n^{1-\alpha/\delta}}\right) \right] = 1_{\{0 < s < t_k\}}(s) \quad (56)$$

To see (56), one takes the limit in regions  $R_{1,k}, \dots, R_{4,k}$ , separately. Fig.3 illustrates the function  $\tilde{f}(\cdot) := f(\frac{\cdot - s}{u/n^{1-\alpha/\delta}})$  over these regions where we consider  $\tilde{f}(t) - \tilde{f}(0)$  as  $n \rightarrow \infty$ . By (28), (54), (55) and (56), we take the limit of (50) and then revert the integration by parts to get the limiting characteristic function as

$$\exp \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} [e^{i \sum_{k=1}^m \xi_k r u 1_{\{0 < s < t_k\}}} - 1 - i \sum_{k=1}^m \xi_k r u 1_{\{0 < s < t_k\}}] \lambda u^{-\delta-1} ds du \gamma(dr).$$

But, this is the characteristic function of  $\sum_{k=1}^m \xi_k Z(t_k)$ , where

$$Z(t) = \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} r u 1_{\{0 < s < t\}} \tilde{N}'(ds, du, dr)$$

for a Poisson random measure  $N'$  with mean measure  $\mu' = \lambda ds u^{-\delta-1} du \gamma(dr)$ , and  $\tilde{N}' = N' - \mu'$ . This characterizes the limiting process by Lemma 3.

To complete the proof of weak convergence, it is sufficient to show that  $\mathbb{E}|Z_n(t) - \mathbb{E}Z_n(t)|^{1+\kappa} \leq C t^b$  for some  $b > 1$  and  $C > 0$  in view of the proof of Theorem 1. In the present theorem, we need a finer estimate given in [38, Lemma 2] and used in [25, Lemma 6]. We have

$$\mathbb{E}|Z_n(t) - \mathbb{E}Z_n(t)|^{1+\kappa} \leq a \int_0^{\infty} (1 - e^{-2I_n}) \xi^{-2-\kappa} d\xi \quad (57)$$

where

$$I_n = \int \int \int \left( 1 - \cos \left( \xi n^{1-\alpha/\delta} r u \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] \right) \right) \mu_n(ds, du, dr)$$

and  $a = (\int_0^\infty (1 - \cos x)x^{-2-\kappa}dx)^{-1}$ , which is finite with  $0 < \kappa \leq 1$ . Substituting  $\mu_n$  and applying integration by parts, we get

$$I_n = \int \int \int \partial_u k(s, u, n^{1-\alpha/\delta} r) \mathbb{P}\{U > nu\} \frac{n^\alpha}{h(n^{\alpha/\delta})} \lambda ds du \gamma(dr) \quad (58)$$

where

$$k(s, u, r) = 1 - \cos \left( \xi r u \left[ f \left( \frac{t-s}{u} \right) - f \left( \frac{-s}{u} \right) \right] \right). \quad (59)$$

For latter use, the partial derivative of  $k$  in  $u$  is found as

$$\begin{aligned} \partial_u k(s, u, r) &= \sin \left( \xi r u \left[ f \left( \frac{t-s}{u} \right) - f \left( \frac{-s}{u} \right) \right] \right) \\ &\cdot \left[ \xi r \left[ f \left( \frac{t-s}{u} \right) - f \left( \frac{-s}{u} \right) \right] + \xi r \left[ f' \left( \frac{t-s}{u} \right) \frac{t-s}{u} - f' \left( \frac{-s}{u} \right) \frac{-s}{u} \right] \right] \end{aligned}$$

Making a change of variable  $u$  to  $u/n^{1-\alpha/\delta}$  in (58), we get

$$I_n = \int \int \int \frac{1}{n^{1-\alpha/\delta}} \partial_u k(s, u/n^{1-\alpha/\delta}, n^{1-\alpha/\delta} r) \mathbb{P}\{U > n^{\alpha/\delta} u\} \frac{n^\alpha}{h(n^{\alpha/\delta})} \lambda ds du \gamma(dr).$$

Note the similarity of  $I_n$  to (50). Moreover, the inequality  $\sin x \leq 2^{1-\kappa}|x|^\kappa \wedge 2$  holds since  $\sin x = [e^{ix} - 1 + (e^{ix} + 1)]/2$  leading to estimates as in (20) and (21). It follows that

$$4M(C + \epsilon) \xi \int \int_0^\infty \int |r| B(s, u, t, n) \max(u^{-\epsilon}, u^\epsilon) u^{-\delta} \lambda ds du \gamma(dr)$$

is an upper bound to  $|I_n|$  when it is evaluated over  $u \geq n_0/n^{\alpha/\delta}$  where  $\epsilon$  and  $n_0$  are as above and

$$\begin{aligned} B(s, u, t, n) &\leq 1_{R_{1,k,n}} + (1 \wedge M|\xi|^\kappa |r|^\kappa u^\kappa) \cdot 1_{R_{2,k,n}} \\ &+ 2 \frac{t-s}{u/n^{1-\alpha/\delta}} 1_{R_{3,k,n}} + \left( 2 \frac{t}{u/n^{1-\alpha/\delta}} + \frac{|s|t}{u^2/n^{2(1-\alpha/\delta)}} \right) 1_{R_{4,k,n}} \end{aligned}$$

For evaluating  $|I_n|$  for smaller values of  $u$ , we have a bound similar to (53). Therefore,  $I_n$  is bounded by an integrable function uniformly over  $n$  in view of the analogous computations in Appendix B. By dominated convergence theorem, let  $I = \lim_n I_n$ . We find that

$$I = \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty [1 - \cos(\xi u r 1_{\{0 < s < t\}})] u^{-\delta-1} \lambda ds du \gamma(dr)$$

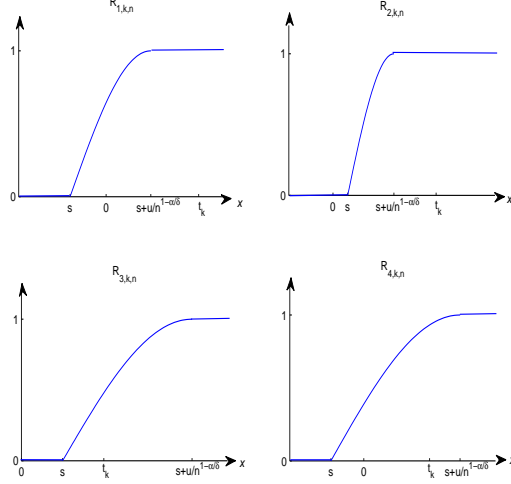


Figure 3: Graphs of  $\tilde{f}(x) := f\left(\frac{x-s}{u/n^{1-\alpha/\delta}}\right)$  in different subregions for  $(s, u/n^{1-\alpha/\delta})$ . The function  $f$  is arbitrary provided that its derivative and itself are Lipschitz continuous almost everywhere.

by using the same approach for taking the limit of the characteristic function of the finite dimensional distributions above. Then, we can write

$$\mathbb{E}|Z_n(t) - \mathbb{E}Z_n(t)|^{1+\kappa} \leq a \int_0^\infty (1 - e^{-4I}) \xi^{-2-\kappa} d\xi \quad (60)$$

for sufficiently large  $n$ , by (57), since  $1 - e^{-x}$  is increasing in  $x$ . Simplifying  $I$  further, we have

$$I = \lambda t \int_{-\infty}^\infty \int_0^\infty [1 - \cos(\xi u r)] u^{-\delta-1} du \gamma(dr) = \lambda t \xi^\delta \mathbb{E}|R|^\delta \int_0^\infty (1 - \cos u) u^{-\delta-1} du$$

where the second equality follows by a change of variable  $u$  to  $u/(\xi|r|)$ . Define the constant  $\tilde{C}$  so that  $I =: \tilde{C}t\xi^\delta$ . Now, substituting  $I$  in (60) and changing  $\xi$  to  $\xi/t^{1/\delta}$ , we get

$$\mathbb{E}|Z_n(t) - \mathbb{E}Z_n(t)|^{1+\kappa} \leq a t^{\frac{1+\kappa}{\delta}} \int_0^\infty (1 - e^{-4\tilde{C}\xi^\delta}) \xi^{-2-\kappa} d\xi$$

which concludes the proof as  $(1 + \kappa)/\delta > 1$ .  $\square$

**Remark 5.** Note that the stable process obtained in the limit is stable with a skewness parameter that depends on the distribution of the rate  $R$ . Moreover, it has

stationary and independent increments. Therefore, it is also a  $\delta$ -stable Lévy motion [33, Def.7.5.1], but with scale parameter

$$\sigma \lambda^{1/\delta} (C_1 + C_2)^{1/\delta}$$

and skewness parameter

$$\beta = \frac{C_1 - C_2}{C_1 + C_2}$$

by [33, pg.s 10,11], where  $C_1 = \mathbb{E}R^\delta 1_{\{R>0\}}$  and  $C_2 = \mathbb{E}|R|^\delta 1_{\{R<0\}}$ , see also [4, pg.217]. In the context of supply and demand, one can interpret  $C_1$  as the skewness caused by demand and  $C_2$  by supply since they are expected to increase and decrease the price, respectively.

**Remark 6.** The weak convergence result given in Theorem 4 is proved with Skorohod's  $J_1$  topology. In [25], the analogous result based on (7) has been omitted. The convergence is shown with  $M_1$  topology instead of  $J_1$  in [36] where the effect function is assumed to be monotone increasing in the context of workload input to the system. The authors heuristically argue that some of the individual loads are too large [36, Rmk.4.2]. On the other hand, [22] proves weak convergence with  $M_1$  topology considering that the limit process has jumps. However,  $J_1$  topology also works as shown above. The interplay between  $J_1$  and  $M_1$  is discussed in [2] for sums of moving averages. It is proved that  $J_1$  convergence cannot hold because adjacent jumps of this process can coalesce in the limit. An intuitive explanation is given as the jump of the limiting process occurring from a staircase of several jumps. Under certain conditions,  $M_1$  convergence is shown instead. We have a simpler situation where each arrival of the scaled process generates a jump of the limit process as evident from (56).

The following theorem is based on the simpler form of the mean measure.

**Theorem 5.** *Suppose the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz continuous with  $f(x) = 0$  for all  $x \leq 0$ ,  $f(x) = f(1) = 1$  for all  $x \geq 1$  and is also differentiable with  $f'$  satisfying a Lipschitz condition a.e., and  $\mathbb{E}|R|^{1+\kappa} < \infty$  for some  $0 < \kappa \leq 1$  with  $1 + \kappa > \delta$ . Let*

$$\tilde{Z}_n(t) = \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} nr u \left[ f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right) \right] \tilde{N}_n(ds, du, dr)$$

where  $\tilde{N} = N - \mu$  and

$$\mu_n(ds, du, dr) = \lambda n^{-\delta} u^{-\delta-1} ds du \gamma(dr).$$

Then, the process  $\{\tilde{Z}_n(t), t \geq 0\}$ , for  $1 < \delta < 2$ , converges in law to

$$(\lambda \mathbb{E}R^\delta 1_{\{R>0\}})^{1/\delta} L_1(t) + (\lambda \mathbb{E}|R|^\delta 1_{\{R<0\}})^{1/\delta} L_2(t)$$

as  $n \rightarrow \infty$ , where  $L_1$  and  $L_2$  are independent  $\delta$ -stable Lévy motions with mean 0, and skewness intensity 1 and  $-1$ , respectively.

**Proof:** We will give only a sketch of the proof due to its similarities with the previous theorem. The characteristic function for the finite dimensional distributions of  $\tilde{Z}$  can be written as

$$\exp \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} g(s, u, nr) \lambda n^{-\delta} u^{-\delta-1} ds du \gamma(dr)$$

with  $g$  as in (16). Making a change of variable  $u$  to  $u/n$ , we get

$$\exp \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} g(s, u/n, nr) \lambda u^{-\delta-1} ds du \gamma(dr) \quad (61)$$

Now,  $g(s, u/n, nr)$  is similar to (55) and we take a similar limit to (56) with  $n^{1-\gamma/\delta}$  replaced by  $n$ . This is justified by dominated convergence theorem since the integrand in (61) can be bounded as in the proof of Theorem 4. Convergence in  $D(0, \infty)$  follows along the same lines, this time with  $k(s, u/n, nr)$  in  $k$  of (59).  $\square$

The simpler form of scalings in Theorems 3 and 5 facilitate neat interpretations in terms of the parameters of the price process. In Theorem 3,  $\lambda$  is scaled as  $n^2\lambda$  and  $r$  is scaled as  $r/n$ , which means that the trading occurs more frequently, but in smaller quantities and yields a fractional Brownian motion limit. In contrast, a stable process is obtained if the rate of trading decreases while its effect rate increases since  $\lambda$  is scaled as  $\lambda/n^\delta$  and  $r$  is scaled as  $nr$  in Theorem 5.

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## Appendix A

We show that the right hand side of (22) is finite. When the right hand side of (22) is splitted over different regions, checking the finiteness of the integrals over  $R_{1,k}, R_{3,k}, R_{4,k}$  reduces to showing that

$$\begin{aligned} \int_{-\infty}^0 \int_{-s}^{t_k-s} u^{-\delta} \max(u^{-\epsilon}, u^{\epsilon}) du ds + \int_0^{t_k} \int_{t_k-s}^{\infty} (t_k-s) u^{-\delta-1} \max(u^{-\epsilon}, u^{\epsilon}) du ds \\ + \int_{-\infty}^0 \int_{t_k-s}^{\infty} \left( \frac{1}{u} + \frac{|s|}{u^2} \right) u^{-\delta} \max(u^{-\epsilon}, u^{\epsilon}) du ds \end{aligned}$$

is finite. This is indeed true when we choose  $\epsilon > 0$  such that

$$1 < \delta - \epsilon < \delta < \delta + \epsilon < 2. \quad (62)$$

In region  $R_{2,k}$ , we have

$$I := \int \int_0^{t_k} \int_0^{t_k-s} |r| [1 \wedge \sum_j M |\xi_j|^{\kappa} |r|^{\kappa} u^{\kappa} 1_{\{s \leq t_j\}}] u^{-\delta} \max(u^{-\epsilon}, u^{\epsilon}) du ds \gamma(dr) \quad (63)$$

If  $t_j > t_k$ , it can be observed from Fig.2 that the integral reduces to that over region  $R_{2,k}$ . If  $t_j < t_k$ , then the integral over  $R_{2,k}$  yields an upper bound. That is, we can replace  $1_{\{s \leq t_j\}}$  by the constant function 1 and get

$$\begin{aligned} I \leq \mathbb{E} |R|^{1+\kappa} \sum_j |\xi_j|^{\kappa} \int_0^{\bar{u}} \int_0^{t_k} u^{\kappa-\delta} \max(u^{-\epsilon}, u^{\epsilon}) ds du \\ + \mathbb{E} |R| \int_{\bar{u}}^{t_k} \int_0^{t_k} u^{-\delta} \max(u^{-\epsilon}, u^{\epsilon}) ds du \end{aligned} \quad (64)$$

where  $\bar{u}$  denotes a cutoff value of  $u$  such that  $\sum_j |\xi_j|^{\kappa} u^{\kappa}$  is too large in (64), and we use the fact that  $t_k - u \leq t_k$  for  $u \geq 0$  after changing the order of integration for  $u$  and  $s$  in (63). Then, the right hand side of (64) is finite if we choose  $\epsilon > 0$  such that

$$1 < \delta - \epsilon < \delta < \delta + \epsilon < 1 + \kappa$$

which clearly satisfies (62) since  $\kappa \leq 1$ .

## Appendix B

In this part, we show that (51) is integrable with respect to  $\max(u^{-\epsilon}, u^\epsilon) u^{-\delta} du ds$ . Substituting the limits of integration in regions  $R_{1,k,n}, R_{3,k,n}, R_{4,k,n}$  shown by  $I_1, I_3, I_4$ , respectively, we have

$$\begin{aligned} I_1 &= \int_{-\infty}^0 \int_{-\tilde{n}s}^{\tilde{n}t_k - \tilde{n}s} \max(u^{-\epsilon}, u^\epsilon) u^{-\delta} du ds \\ I_3 &= \int_0^{t_k} \int_{\tilde{n}t_k - \tilde{n}s}^{\infty} (\tilde{n}t_k - \tilde{n}s) \max(u^{-\epsilon}, u^\epsilon) u^{-\delta-1} du ds \\ I_4 &= \int_{-\infty}^0 \int_{\tilde{n}t_k - \tilde{n}s}^{\infty} \left( \frac{2}{u/\tilde{n}} + \frac{|s|}{u^2/\tilde{n}^2} \right) \max(u^{-\epsilon}, u^\epsilon) u^{-\delta} du ds \end{aligned}$$

where we put  $\tilde{n} = n^{1-\alpha/\delta}$ . The integrals  $I_1, I_3, I_4$  are finite for

$$1 < \delta - \epsilon < \delta < \delta + \epsilon < 2$$

since

$$\begin{aligned} \int_{-\infty}^0 \int_{-\tilde{n}s}^{\tilde{n}t_k - \tilde{n}s} u^{-\tilde{\delta}} du ds &= C_1 \frac{t_k^{2-\tilde{\delta}}}{\tilde{n}^{\tilde{\delta}-1}} \leq C_1 t_k^{2-\tilde{\delta}} \\ \int_0^{t_k} \int_{\tilde{n}t_k - \tilde{n}s}^{\infty} (\tilde{n}t_k - \tilde{n}s) u^{-\tilde{\delta}-1} du ds &= C_2 \frac{t_k^{2-\tilde{\delta}}}{\tilde{n}^{\tilde{\delta}-1}} \leq C_2 t_k^{2-\tilde{\delta}} \\ \int_{-\infty}^0 \int_{\tilde{n}t_k - \tilde{n}s}^{\infty} \left( \frac{2}{u/\tilde{n}} + \frac{|s|}{u^2/\tilde{n}^2} \right) u^{-\tilde{\delta}} du ds &\leq C_3 \frac{t_k^{2-\tilde{\delta}} + t_k^{-\tilde{\delta}} + t_k^{1-\tilde{\delta}}}{\tilde{n}^{\tilde{\delta}-1}} \\ &\leq C_3 (t_k^{2-\tilde{\delta}} + t_k^{-\tilde{\delta}} + t_k^{1-\tilde{\delta}}) \end{aligned}$$

for  $1 < \tilde{\delta} < 2$  and  $\tilde{n} \geq 1$ , where  $C_1, C_2, C_3 \in \mathbb{R}$ . In  $R_{2,k,n}$ , we have

$$I_2 = \int_0^{t_k} \int_0^{\tilde{n}t - \tilde{n}s} \left( 1 \wedge \sum_j M |\xi_j|^\kappa |r|^\kappa u^\kappa 1_{\{s \leq t_j\}} \right) \max(u^{-\epsilon}, u^\epsilon) u^{-\delta} du ds$$

As in Appendix A, we consider two intervals  $[0, \bar{u}]$  and  $(\bar{u}, t_k]$  to evaluate this integral. Over the first interval, it is finite for  $1 < \tilde{\delta} < 1 + \kappa$ , and over the latter, it is proportional to  $\tilde{n}^{1-\tilde{\delta}}$  which is bounded by 1. As a result, (51) is finite if we choose  $\epsilon > 0$  such that

$$1 < \delta - \epsilon < \delta < \delta + \epsilon < 1 + \kappa.$$